ISSN 0202-2893, Gravitation and Cosmology, 2011, Vol. 17, No. 3, pp. 201–217. © Pleiades Publishing, Ltd., 2011.

A Topological Interpretation of Quantum Theory and Elementary Particle Structure

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Received July 20, 2010; in final form, February 22, 2011

Abstract—We present a new concept of topological and geometric interpretation of quantum mechanics. A special choice of geometric markers makes it possible to connect quantum mechanics with a topological interpretation of the electric charge and to build an electrodynamics with integer-valued point charges. The electric charge gains the status of a topological charge in the form of a geometrically distinguished region of the physical space with nonzero curvature. We introduce a topological interpretation of particles and compare it with elementary particle properties. A topological interpretation of the baryonic charge is suggested.

DOI: 10.1134/S0202289311030121

1. INTRODUCTION

The basic problems faced by electrodynamics, both classical and quantum, are reduced to the following fundamental facts. The first one is that the electrons and other elementary particles are defined in quantum theory from the very beginning as point objects. It is sufficient to recall the formulation of the statistical postulate that the squared absolute value of the wave function is connected with the probability density of finding the particle at a given point of space at given time. Thus the postulate unambiguously treats the particle as a point object. Endowed with electric charges, such objects create around them a Coulomb electric field whose strength and potential tend to infinity at the charge location. This in turn leads to an infinite energy of the field of such charges. A simple solution of this problem, sometimes considered and related to introducing particles of finite size, allows for obtaining finite values of the field strength and energy. However, this not only contradicts the postulated pointlike nature of the particles but also the experimental fact that particle charges are multiples of the electron charge. For extended objects there are no non-exotic models in which an extended charge has always a value being a strict multiple of the electron charge.

One of the remarkable ideas, that of a “charge without charge”, allowing for avoidance of the Coulomb energy divergence, has been suggested by Wheeler and Misner [1–3]. The idea consists in using regions with a complicated topology (Wheeler handles) in which point charges are absent although for an external observer such a region of space would look as a pair of objects with opposite charges. The regions with such topologies possess a significant curvature, which, from the viewpoint of general relativity (GR), implies the existence of large energy densities that determines the mass of such objects. Although this idea has quite a number of shortcomings, it has still laid a basis for the topological approach to a possible formulation of the problems of quantum particle theory (see, e.g., [4–6]). It is for this reason that topological charges are one of the important elements of quantum field theory [7].

The idea due to Wheeler and Misner has grown from an analysis of possible consequences of GR at small scales where the dynamics obeys the laws of quantum theory. This range of scales was considered in GR from the beginning as a separate problem. However, as quantum theory was developing, it became clear that at the energy densities inherent to the structure of nucleons, the effects of space-time curvature must appear at small scales. The problem of reconciling the viewpoints of GR and quantum theory at the level of particle structure still faces significant difficulties and still remains unsolved. A possible explanation of these difficulties of GR may be connected with a complex space-time topology at small scales, and one can consider the problem of reconciling GR with the quantum laws on the basis of their interpretation as manifestations of this complicated topology. In this approach, the large-scale theory of gravity, based on the idea of GR on a non-Euclidean nature of space-time, should be a continuation of quantum laws in their new topological

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interpretation in the form of a "residual" space-time curvature at large scale.

Following the general idea of invoking space-time topology for explaining the particle properties, a new approach to the description of particle structure was suggested in [8–10], based on a topological interpretation of the electric charge and other charge numbers such as the baryonic charge. The basic meaning of the postulates introduced in these papers consists in a simple topological interpretation of particle structure as elements of the geometric structure of the physical space introduced as a hypersurface in the space of dimension 4. A particle is defined as a region of the three-dimensional physical space bounded by a closed two-dimensional orientable surface. All such two-dimensional surfaces are classified with the aid of the well-known topological theorem (see, e.g., [11, 12]) as a sphere with g handles glued into it. As a result, it becomes possible to completely describe all particle types in a topological manner. The basic meaning of such an approach is that in its framework the particle classification looks natural and corresponds to the main charge properties of the particles.

One more important element of this approach is the opportunity of obtaining a new topological and geometric interpretation of the quantum laws. In this respect, the theory resembles a theory with hidden parameters. However, the hidden parameters appearing in this work have another nature than the parameters described by von Neumann's well-known theorem [13]. The meaning of the new hidden variables is their essential nonlocality, while those in von Neumann's theorem are local. The existence of theories with nonlocal hidden parameters was pointed out earlier (see [14]).

The action of the introduced nonlocal (topological) parameters can be illustrated in the following way. A topological restructuring can be a result of a local change in the structure of space. For example, a continuous mutual approach of two maxima of a function which describes the properties of space in a given theory does not change its topological properties until they merge. When they merge, one of the maxima disappears, maybe along with a saddle point between them. However, such a local structure change leads at once to a global topological restructuring of space, i.e., a change in its global topological properties. At the micro-level, it should look like an instantaneous change in the quantum properties of the system. Therefore a collapse of the wave function is connected in this theory with topological restructurings of the system at the instant of measurement. As a result, this element of the approach being suggested makes it possible to give a "rational" explanation of the projection postulate, the collapse of a wave function when a measurement is carried out. Since in a quantum experiment we always obtain averaged parameters of systems, the topological restructurings look as instantaneous shrinks of wave functions to the revealed state.

In the present paper, this approach is developed and improved. We derive the equations of electrodynamics with integer-valued point charges. One of the main goals of this work was to build a topological interpretation of the electric charge along with its foundation in the framework of quantum theory. In the first sections of the paper we carry out a derivation of the Schrödinger equation which allows us to consider the topological structures obtained as objects of quantum theory. One should note that geometric considerations were invoked for foundation of the quantum laws in some previous papers (see, e.g., [15] and references therein).

In this paper, we suggest a new approach to the interpretation of quantum theory on the basis of geometric Lagrangian markers. This makes possible an introduction of the Lagrangian marker density in a natural way, and using it, we construct an averaged description of particle dynamics and the related Schrödinger equation for charges particles.

2. THE BACKGROUND GEOMETRY AND PHYSICAL SPACE

To solve the problems related to the properties of point charges, consider the following construction. Suppose that at small distances near a point charge the space has a certain curvature and is thus non-Euclidean. Meanwhile, the local increase in curvature at small scale is perceived by us as an increased matter energy density near that point whereas our ideas on the metric properties of the space itself remain to be connected with the "background" Euclidean space P^3 of dimension $d = 3$ that surrounds us. As a result, our view of the nature of spatial changes of the physical quantities near a point charge correspond to the background space rather than the real physical space. For clarity we can try to describe this situation in terms of the external geometry. Suppose that our physical space V^3 is a 3-dimensional surface in the ambient Euclidean space W^4 with the dimension $d = 4$. Such a hypersurface may be specified using the height function $\mathcal{F} = \mathcal{F}(\mathbf{x}, t)$, i.e., by the algebraic equation

$$u = \mathcal{F}(\mathbf{x}, t), \qquad (1)$$

where u is the extra fourth coordinate in the ambient space $W^4 = T_u \times P^3$, and $\mathbf{x} = (x, y, z)$ are Cartesian coordinates in the hyperplane perpendicular to the direction u in the ambient space which is considered as the background Euclidean space. We will temporarily assume that motions of the object under consideration occur at such velocities that one can neglect the relativistic effects. Therefore the time t is absolute.

To make more precise the physical meaning of this construction, we note that, in the usual sense, the space is just the "ambient" space of dimension 4 (or the space-time of dimension 5). The words "in the usual sense" mean here that the space is considered as an arena on which manifests itself the action of physical laws. This space is assumed to be Euclidean. And it is with this space that we connect our inertial reference frames, i.e., reference frames attached to the remote objects (stars, galaxies, etc.). The space observed as the physical space is a certain curved hypersurface in this ambient space. Singling out such a hypersurface from a physical viewpoint can only be related to its "material nature". There is so far no necessity to consider or explain the meaning of this material nature (of what it consists) because at the present stage only its geometric and topological structure is important.

Assume at the physical hypersurface V^3 a distinguished curvilinear coordinate system $\mathbf{e} = (e^1, e^2, e^3)$, which is in some way connected with the physically significant elements of the external geometry of V^3 as a hypersurface in W^4; their meaning will be made more precise in the theory construction process. Since the coordinate system $\mathbf{e}$ is physically distinguished, the mapping $\mathbf{x} \to \mathbf{e}$ should be defined with the aid of the functions

$$e^a = e^a(\mathbf{x}, t).$$

It means that a displacement of elements of the external geometry (and accordingly the internal one) with respect to the background space P^3 will be perceived as a change of the functions $e^a(\mathbf{x}, t)$ in space and time. A question emerges: in which way could one connect the general changes of the coordinates e^a with the physical characteristics corresponding to the observed geometric characteristics of the physical space, i.e., is it possible to endow the functions $e^a(\mathbf{x}, t)$ or related quantities with a physical meaning?

3. CHANGES OF THE CURVILINEAR COORDINATES IN TIME

To make clear the possible physical meaning of the functions $e^a(\mathbf{x}, t)$, consider a vector field $\mathbf{V}(\mathbf{x}, t)$ with the components $V^\alpha = V^\alpha(\mathbf{x}, t)$, $\alpha = 1, 2, 3$ in its projection onto the background space P^3, satisfying the following set of equations:

$$\frac{\partial e^a}{\partial t} + V^\beta \frac{\partial e^a}{\partial x^\beta} = 0, \quad a = 1, 2, 3. \tag{2}$$

The field $\mathbf{V}$ is the field of transfer of the curvilinear coordinates relative to the Cartesian coordinate system (reference frame) $\mathbf{x}$ of the background space. Eq. (2), at all points where the coordinate transformation $\Im$: $\mathbf{x} \to \mathbf{e}(\mathbf{x}, t)$ is not degenerate, makes it possible to calculate the field $\mathbf{V}$ unambiguously in terms of the functions $e^a(\mathbf{x}, t)$:

$$V^\alpha = -\frac{\partial x^\beta}{\partial e^a} \frac{\partial e^a}{\partial t}.$$

The condition that the mapping $\Im$ is not degenerate is equivalent to the condition that the Jacobian of this transformation

$$J = \det\left(\partial e^a / \partial x^\alpha\right) \tag{3}$$

is nonzero. Let us note that the field $\mathbf{V}$ is similar in meaning to the hydrodynamic field of Euler transfer velocities of Lagrangian markers connected with the coordinates e^a. One can actually assert that, in our approach, the coordinates e^a are Lagrangian markers of some geometric characteristics of the physical space projected onto the Euclidean background space.

From (2) we can obtain an equation for the Jacobian J. Differentiating (2) with respect to the coordinates $\mathbf{x}$, we arrive at the relations

$$\frac{\partial}{\partial t} \frac{\partial e^a}{\partial x^\alpha} + \frac{\partial}{\partial x^\alpha} \left(V^\beta \frac{\partial e^a}{\partial x^\beta} \right) = 0.$$

Hence it follows

$$\frac{\partial x^\alpha}{\partial e^a} \frac{\partial}{\partial t} \frac{\partial e^a}{\partial x^\alpha} + \frac{\partial x^\alpha}{\partial e^a} \frac{\partial}{\partial x^\alpha} \left(V^\beta \frac{\partial e^a}{\partial x^\beta} \right) = 0.$$

Since the matrices $J^\alpha_a = \partial x^\alpha / \partial e^a$ and $J^a_\alpha = \partial e^a / \partial x^\alpha$ are mutually inverse, we have the following identities:

$$\frac{\partial x^\alpha}{\partial e^a} \frac{\partial}{\partial t} \frac{\partial e^a}{\partial x^\alpha} = \frac{1}{|J|} \frac{\partial |J|}{\partial t},$$

$$\frac{\partial x^\alpha}{\partial e^a} \frac{\partial}{\partial x^\alpha} \left(V^\beta \frac{\partial e^a}{\partial x^\beta} \right) = V^\beta \frac{1}{|J|} \frac{\partial |J|}{\partial x^\beta} + \frac{\partial}{\partial x^\beta} V^\beta.$$

Therefore we obtain an equation for the Jacobian J:

$$\frac{\partial}{\partial t} |J| + \frac{\partial}{\partial x^\beta} \left(V^\beta |J| \right) = 0. \tag{4}$$

This equation describes the time dependence of J. The geometric and physical meaning of this equation is that in any domain V_0 of the physical space, whose boundary is specified by equations in the coordinates $\mathbf{e}$ fixed in time and whose image in the background space is $\mathcal{V}_0$, holds the conservation law

$$\int\limits_{\mathcal{V}_0} |J| dx\, dy\, dz = \int\limits_{\mathcal{D}_0} de^1 de^2 de^3 = |\mathcal{V}_0| = \text{const.} \tag{5}$$

Here $\mathcal{D}_0$ is the image of the domain $\mathcal{V}_0$ in the Cartesian coordinate patch $\mathbf{e}$ while $|\mathcal{V}_0|$ is the volume of the domain $\mathcal{D}_0$ corresponding to $\mathcal{V}_0$.

Consider domains of the space V^3 for which the condition (5) holds. We will call this condition the normalization condition. On P^3, the images of boundaries of such domains, in hydrodynamic terms, are rigidly connected with the Lagrangian markers e^a. In such domains the function $|J|$ is the conserved density of markers due to (4) and (5). Since we suppose that the markers are related to some geometric properties of the physical space V^3, the quantity $|J|$ may be interpreted as the density of the physical space with respect to the flat background space. Thus it becomes possible to consider a set of integral characteristics of domains V_0 which can be interpreted as dynamic properties of particles. On each domain of this kind, one can connect the Jacobian J with an invariant averaging operation for physical quantities $Q(\mathbf{x}, t)$ connected with the background space and defined on these domains with the aid of the relations

$$\overline{Q} = \int\limits_{\mathcal{V}_0} Q(\mathbf{x}, t)\widetilde{J} d\mathcal{V}, \qquad (6)$$

where

$$\widetilde{J} = \frac{1}{|V_0|}|J|, \qquad \int\limits_{\mathcal{V}_0} \widetilde{J} d\mathcal{V} = 1. \qquad (7)$$

We have, in particular,

$$X^\alpha = \overline{x}^\alpha = \int\limits_{\mathcal{V}_i} x^\alpha \widetilde{J} dV. \qquad (8)$$

The quantities $\mathbf{X} = \overline{\mathbf{x}} = (\overline{x}^1, \overline{x}^2, \overline{x}^3)$ may be considered as the mean coordinates of a "particle" moving in the domain $\mathcal{V}_0$ and together with it. The dynamics of such motion can be described in terms of averaged equations of motion. By definition, the velocity of particle motion has the following form:

$$U^\alpha(t) = \frac{dX^\alpha}{dt} = \frac{d}{dt}\overline{x}^\alpha = \int\limits_{\mathcal{V}_0} x^\alpha \frac{\partial \widetilde{J}}{\partial t} d\mathcal{V}$$
$$= -\int\limits_{\mathcal{V}_0} x^\alpha \mathrm{div}\,(\widetilde{J}\mathbf{V}) d\mathcal{V} + \int\limits_{\partial\mathcal{V}_0} x^\alpha v^\beta d\sigma_\beta.$$

Here we have used Eq. (4) and the condition that points of the boundary of the domain $\mathcal{V}_0$ move as images of points of the fixed domain V_0 in V^3. Therefore the field $\mathbf{v}$ with the components v_β coincides with the velocity $\mathbf{V}$ on the edge of the domain. Calculating the last integral by parts, we find:

$$\overline{V}^\alpha = \frac{d}{dt}\overline{x}^\alpha = \int\limits_{\mathcal{V}_0} V^\alpha d\mathcal{V} + \int\limits_{\partial\mathcal{V}_0} x^\alpha (v^\beta - V^\beta) d\sigma_\beta.$$

Putting $v^\beta = V^\beta$, which means that the changing velocity of the domain boundaries is described by the field V^β, we arrive at the final equality

$$U^\alpha = \frac{d}{dt}\overline{x}^\alpha = \frac{dX^\alpha}{dt} = \int\limits_{\mathcal{V}_0} V^\alpha d\mathcal{V}. \qquad (9)$$

Consequently, the mean velocity of "particle" motion is equal to the averaged transport velocity of the curvilinear coordinates, i.e., of the geometric properties that distinguish the particle as a spatial domain. In a similar way we calculate the particle acceleration:

$$\frac{d}{dt}U^\alpha = \int\limits_{\mathcal{V}_0} \left(\frac{\partial V^\alpha}{\partial t}\widetilde{J} + V^\alpha \frac{\partial \widetilde{J}}{\partial t}\right) d\mathcal{V} + \int\limits_{\partial\mathcal{V}_0} V^\alpha v^\beta d\sigma_\beta$$
$$= \int\limits_{\mathcal{V}_0} \left(\frac{\partial V^\alpha}{\partial t}\widetilde{J} - V^\alpha \mathrm{div}\,(\widetilde{J}\mathbf{V})\right) dV + \int\limits_{\partial\mathcal{V}_0} V^\alpha v^\beta d\sigma_\beta$$
$$= \int\limits_{\mathcal{V}_0} \left(\frac{\partial V^\alpha}{\partial t} + V^\beta \frac{\partial V^\alpha}{\partial x^\beta}\right) \widetilde{J} d\mathcal{V}$$
$$+ \int\limits_{\partial\mathcal{V}_0} V^\alpha (v^\beta - V^\beta) d\sigma_\beta.$$

We finally obtain

$$\frac{dU^\alpha}{dt} = \frac{d^2 X^\alpha}{dt^2} = \int\limits_{\mathcal{V}_i} \left(\frac{\partial V^\alpha}{\partial t} + V^\beta \frac{\partial V^\alpha}{\partial x^\beta}\right) \widetilde{J} dV. \qquad (10)$$

Thus the mean acceleration of a "particle" as a spatial domain is equal to the average Eulerian acceleration of the flow connected with $\mathbf{V}$.

4. THE CLASSICAL EQUATIONS OF AVERAGED DYNAMICS

The relations (10) can be considered as the Newtonian equations for a "particle" connected with the domain $\mathcal{V}_0$, whose dynamics is determined by the averaged analytic properties of the hydrodynamic flow $\mathbf{V}$. This interpretation will be justified if we are able to show that the analytic properties of the flow $\mathbf{V}$ can be interpreted in terms of the known force fields, namely, the electromagnetic and maybe gravitational ones. To do so, consider the following identity for the Eulerian acceleration known in hydrodynamics:

$$\frac{dV^\alpha}{dt} \equiv \frac{\partial V^\alpha}{\partial t} + V^\beta \frac{\partial V^\alpha}{\partial x^\beta}$$
$$= \frac{1}{2}\frac{\partial |\mathbf{V}|^2}{\partial x^\alpha} - \Big[\mathbf{V} \times \mathrm{rot}\,\mathbf{V}\Big] + \frac{\partial V^\alpha}{\partial t}. \qquad (11)$$

Let us formally present the field $\mathbf{V}$ as a sum of two terms:

$$\mathbf{V} = -\Gamma_0 \mathbf{A} + \boldsymbol{\nabla}\chi. \qquad (12)$$

The factor Γ_0 has been introduced for convenience of the subsequent interpretation. Substituting (12) into the identity (11), we arrive at its representation as follows:

$$\frac{dV^\alpha}{dt} \equiv \frac{\partial V^\alpha}{\partial t} + V^\beta \frac{\partial V^\alpha}{\partial x^\beta}$$
$$= \frac{\partial}{\partial x^\alpha}\left(\frac{1}{2}|\mathbf{V}|^2 + \frac{\partial \chi}{\partial t} + \Gamma_0 \Phi\right)$$
$$+ \Gamma_0 \Big[\mathbf{V} \times \mathrm{rot}\,\mathbf{A}\Big]^\alpha - \Gamma_0 \left(\frac{\partial A^\alpha}{\partial t} + \frac{\partial \Phi}{\partial x^\alpha}\right). \quad (13)$$

Here we have introduced the additional function Φ which allows for formally identifying the elements of this representation as the parameters of a certain electromagnetic field in the following way:

$$E^\alpha = -\frac{\partial A^\alpha}{\partial t} - \frac{\partial \Phi}{\partial x^\alpha}, \qquad B^\alpha = \mathrm{rot}\,\mathbf{A}.$$

In this case, the identity (13) may be interpreted, again formally, as the Euler equations for the dynamics of a charged fluid in a magnetic field with the induction $\mathbf{B}$ and an electric field with the strength bE. Using the averaging procedure, we can now introduce a mean (macroscopic) electromagnetic field with the parameters

$$\overline{\mathbf{A}} = \int\limits_{\mathcal{V}_i} \mathbf{A}\widetilde{J}dV, \quad \overline{\mathbf{B}} = \int\limits_{\mathcal{V}_i} \mathbf{B}\widetilde{J}dV, \quad \overline{\mathbf{E}} = \int\limits_{\mathcal{V}_i} \mathbf{E}\widetilde{J}dV.$$

This allows us to present the "microscopic" fields $\mathbf{A}$, $\mathbf{B}$, $\mathbf{E}$ in the following way:

$$\mathbf{A} = \overline{\mathbf{A}} + \mathbf{A}', \qquad \mathbf{B} = \overline{\mathbf{B}} + \mathbf{B}', \qquad \mathbf{E} = \overline{\mathbf{E}} + \mathbf{E}'.$$

Here $\mathbf{A}'$, $\mathbf{B}'$, $\mathbf{E}'$ are fields with zero mean values. As a result, the formula for the mean acceleration takes the form

$$\frac{d}{dt}\overline{\mathbf{V}} = \Gamma_0 \overline{\mathbf{E}} - \Gamma_0 \Big[\overline{\mathbf{V}} \times \overline{\mathbf{B}}\Big] - \boldsymbol{\nabla}_X \overline{U} + \mathbf{F}_q. \quad (14)$$

This relation looks as the Newtonian equation for a charged particle in the electromagnetic field $\{\overline{\mathbf{B}}, \overline{\mathbf{E}}\}$ and an additional conservative field with the potential $\overline{\mathbf{U}}$:

$$\overline{\mathbf{U}}(\mathbf{X}, t) = \sum_{|\mathbf{k}|=0} \frac{1}{|\mathbf{k}!|} \frac{\partial^{|k|} U(\mathbf{X})}{\partial X_1^{k_1} \partial X_2^{k_2} \partial X_3^{k_3}} M_{\mathbf{k}}(t).$$

Here,

$$U(\mathbf{x}, t) = \frac{1}{2}|\mathbf{V}|^2 + \chi_t + \Gamma_0 \Phi, \quad (15)$$

$\mathbf{k} = (k_1, k_2, k_3)$ is a multiindex, $|\mathbf{k}| = k_1 + k_2 + k_3$, $|\mathbf{k}!| = k_1!k_2!k_3!$, and

$$M_{\mathbf{k}} = M_{k_1 k_2 k_3}(t)$$
$$= \int\limits_{\mathcal{V}_i} (x^1 - X^1)^{k_1} (x^2 - X^2)^{k_2} (x^3 - X^3)^{k_3} \widetilde{J}dV.$$

The additional force $\mathbf{F}_q$, having the form

$$F_q^\alpha = \int\limits_{\mathcal{V}_i} \Gamma_0 \Big[\mathbf{V}' \times \mathbf{B}'\Big]^\alpha \widetilde{J}dV,$$

can be considered as a fluctuation term, similar to the additional terms emerging in quantum theory in derivations of averaged equations from operator equations.

Following this general ideology, we can also indicate the general form of the integrals of motion for the set of Newtonian equations. According to (6), for a conserved quantity $\overline{Q}$ we have

$$\frac{d\overline{Q}}{dt} = \int\limits_{\mathcal{V}_i} \left(\frac{\partial Q}{\partial t} + V^\alpha \frac{\partial Q}{\partial x^\alpha}\right) \widetilde{J}dV = 0.$$

Hence it follows that conserved are all quantities $\overline{Q}$ which have "microscopic" images $Q(\mathbf{x}, t)$ satisfying the equation

$$\frac{\partial Q}{\partial t} + V^\alpha \frac{\partial Q}{\partial x^\alpha} = 0.$$

By (2), all such functions must have the following form:

$$Q(\mathbf{x}, t) = \mathcal{Q}\Big(e^1(\mathbf{x}, t), e^2(\mathbf{x}, t), e^3(\mathbf{x}, t)\Big),$$

i.e., must be functions of the coordinates $\mathbf{e}$ only.

These constructions demonstrate that, in the framework of the suggested approach, the classical mechanics of charged particles moving in electric and magnetic fields is reproduced completely. However, to justify it from the viewpoint of elementary particle theory, it is necessary to show that, in the framework of the same approach, it is also possible to reproduce the fundamentals of quantum theory.

5. THE SCHRÖDINGER EQUATION

Consider the relation (15) not as a definition of the function U but as an equation for the function χ:

$$\chi_t + \frac{1}{2}\Big(|\boldsymbol{\nabla}\chi|^2 + 2\Gamma_0(\mathbf{A}, \boldsymbol{\nabla}\chi) + \Gamma_0^2|\mathbf{A}|^2\Big)$$
$$+ \Gamma_0 \Phi - U = 0. \quad (16)$$

This relation is equivalent by its form to the Hamilton-Jacobi equation for a charged particle in an electromagnetic field with the vector potential $\mathbf{A}$, the electric potential Φ and some additional potential U, and it is necessary to choose the parameter Γ_0 in the following way: $\Gamma_0 = e/mc$, where e is the electron charge, m its mass, and c the speed of light. As is well known, the Hamilton-Jacobi equation is connected with the Schrödinger equation. To establish this connection, let us present the function U as a sum of two terms:

$$U = \frac{\hbar^2}{2}\frac{\Delta|J|}{|J|} + U_G.$$

As a result, the relation (16) acquires the form

$$\chi_t + \frac{1}{2}\Big(|\boldsymbol{\nabla}\chi|^2 + 2\Gamma_0(\mathbf{A},\boldsymbol{\nabla}\chi) + \Gamma_0^2|\mathbf{A}|^2\Big) + \Gamma_0\Phi - \frac{\hbar^2}{2}\frac{\Delta|J|}{|J|} + U_G = 0.$$

Unifying the latter equation with Eq. (4), by a direct inspection we verify that the function

$$\psi = \sqrt{|J|}e^{i\chi/\hbar} \tag{17}$$

satisfies the Schrödinger equation for a charged particle in an electromagnetic field:

$$i\hbar\frac{\partial\psi}{\partial t} = \frac{1}{2}\Big(-i\hbar\boldsymbol{\nabla} - \Gamma_0\mathbf{A}\Big)^2\psi + \Big(\Gamma_0\Phi - U_G\Big)\psi = 0. \tag{18}$$

In this equation, the only element that causes a certain difficulty with interpretation is the potential U_G. Since this equation so far does not contain elements which could be related to nuclear forces (weak and strong), the only reasonable interpretation of this potential is to relate it to gravitational forces. However, this question requires a further study.

Since the interpretation of all terms in Eq. (18) does not represent any difficulty (after the above comment on the potential U_G), this equation can really be considered as an equation for a single quantum particle moving in the region $\mathcal{V}_0$, which, as a matter of fact, reflects the changes in local geometric characteristics of the physical space in the corresponding domain of the physical space V_0. Indeed, the squared absolute value of the wave function is equal to the Jacobian of the mapping $\Im$, which can be interpreted as a geometric equivalent of the statistical postulate. Then the established connection between the averaged dynamics of such particles in the form of the usual Newton or Schrödinger equations provides the validity of the correspondence principle and the Ehrenfest theorem. The linearity of the Schrödinger equation provides the validity of the superposition principle, to which, in the framework of the present approach, it is necessary to give a geometric interpretation.

6. THE GEOMETRIC MARKER AND THE COULOMB LAW IN CURVILINEAR COORDINATES

To make the obtained model of quantum mechanics really complete, it is necessary to include into consideration particle systems and to point out explicitly the geometric characteristics of the physical space, whose markers are the coordinates e^a, in a real physical situation dealt with in quantum experiments.

Let there be, on the background hypersurface P^3 and in a neighborhood of the physical hypersurface V^3, a function $\mathcal{F}$. We will suppose that $\mathcal{F}$ is a Morse function [16], i.e., it is smooth and all its critical points are isolated. In what follows we will call this function the fundamental potential. Let us relate to the function $\mathcal{F}$ a distinguished coordinate system with the aid of the following equations:

$$\frac{\partial x^\beta}{\partial e^a}\frac{\partial\mathcal{F}}{\partial x^\beta} = \varepsilon e^a, \qquad a = 1,2,3. \tag{19}$$

Here $\varepsilon = \pm1$ depending on whether $\mathcal{F}$ achieves a maximum or a minimum at the point $P_0(\mathbf{x}_0, t)$ with the coordinates $\mathbf{x}_0$, which is the image of the origin of e^a, i.e., the point at which $e^a(\mathbf{x}_0, t) = 0$. That means

$$\varepsilon = \operatorname{signdet}\left(\frac{\partial^2\mathcal{F}}{\partial x^\alpha\partial x^\beta}\right)\bigg|_{P_0}. \tag{20}$$

That $\mathcal{F}$ achieves a maximum or a minimum at P_0 follows from (19). Indeed, (19) implies

$$\frac{\partial\mathcal{F}}{\partial x^\beta} = \varepsilon\frac{\partial e^a}{\partial x^\beta}e^a = \varepsilon\frac{\partial|\mathbf{e}|^2}{\partial x^\beta}.$$

Hence we find that near the point P_0 the function $\mathcal{F}$ has the form

$$\mathcal{F} = \varepsilon|\mathbf{e}(\mathbf{x})|^2/2 + \mathcal{F}_0, \tag{21}$$

where $\mathcal{F}_0$ is an integration constant. Since at the point $P_0(\mathbf{x}_0)$ we have $\mathbf{e}(\mathbf{x}_0) = 0$, we find that $\mathcal{F}_0 = \mathcal{F}(\mathbf{x}_0)$, and the function $\mathcal{F}(\mathbf{x})$ at the point $\mathbf{x}_0$ reaches a local extremum, a minimum or a maximum depending on the sign ε, whence follows the relation (20).

To make clear the physical meaning of the choice of the set of equations (19) or it equivalent (21) as a connection between the function $\mathcal{F}$ and $e^a(\mathbf{x}, t)$, we will show, following [8–10], that with this choice the function $\mathcal{F}$ is related in a natural way to a field structure equivalent to a system of point integer-valued charges whose dynamics is described by the Maxwell equations.

Consider, in the Cartesian coordinate system $\mathbf{e}$, the following identity:

$$\frac{\partial}{\partial e^a}\left(\frac{e^a}{|\mathbf{e}|^3}\right) = 4\pi\delta(\mathbf{e}). \tag{22}$$

Being transformed from the coordinates $\mathbf{e}$ to the "background" Cartesian coordinates $\mathbf{x}$, this identity acquires the form

$$\frac{1}{|J|}\frac{\partial}{\partial x^\alpha}\left(|J|\frac{E^\alpha}{|\mathbf{e}(\mathbf{x})|^3}\right) = 4\pi\delta(\mathbf{e}(\mathbf{x})). \tag{23}$$

Here $|\mathbf{e}|^2 = e^a e^a = (e^1)^2 + (e^2)^2 + (e^3)^2$ is the squared length of the radius vector $\mathbf{e}$ in the Cartesian map,

$$E^\alpha = \frac{\partial x^\alpha}{\partial e^a} e^a,$$

and J is the Jacobian of the coordinate transformation $\mathbf{e} \to \mathbf{x}$:

$$J = \det\left(\frac{\partial e^a}{\partial x^\alpha}\right).$$

The field $\mathbf{E}$ with the components E^α can be calculated in the following way. For e^a we have the identity

$$e^a = \frac{1}{2}\frac{\partial |\mathbf{e}|^2}{\partial e^a},$$

From this identity and the condition (19) we find:

$$E_\alpha = \frac{\partial e^a}{\partial x^\alpha} e^a = \frac{1}{2}\frac{\partial e^a}{\partial x^\alpha}\frac{\partial |\mathbf{e}|^2}{\partial e^a} = \frac{1}{2}\frac{\partial |\mathbf{e}|^2}{\partial x^\alpha} = \varepsilon\frac{\partial \mathcal{F}}{\partial x^\alpha}. \quad (24)$$

Since by definition $\mathbf{x}$ are Cartesian coordinates, for these coordinates there is no distinction between upper and lower indices. Therefore we have

$$E^\alpha = E_\alpha = \frac{1}{2}\frac{\partial |\mathbf{e}|^2}{\partial x^\alpha} = \varepsilon\frac{\partial \mathcal{F}}{\partial x^\alpha}. \quad (25)$$

Thus $\mathbf{E}$ is a potential field, with the potential equal to $\mathcal{F} = \varepsilon|\mathbf{e}(\mathbf{x})|^2/2 + \mathcal{F}_0$, and turns to zero at the location of the point charge. Meanwhile, the field $\mathbf{D}$ from (23),

$$\mathbf{D} = \frac{|J|}{|\mathbf{e}|^3}\boldsymbol{\nabla}\mathcal{F}, \quad (26)$$

has a Coulomb singularity with the sign of the charge determined by the parameter ε. It is easily verified under the requirement formulated above that the functions $\mathbf{e} = \mathbf{e}(\mathbf{x})$ are smooth near the point $\mathbf{e} = 0$. Due to (21),

$$|\mathbf{e}|^2 = 2|\mathcal{F} - \mathcal{F}_0|. \quad (27)$$

This relation indicates the geometric meaning of the mapping $\Im: \mathbf{x} \to \mathbf{e}$. It implies that each isosurface of the function $\mathcal{F}$, corresponding to a value $\mathcal{F}_1$ of this function, in some neighborhood of the point P_0 maps into a two-dimensional sphere of radius $R = 2|\mathcal{F}_1 - \mathcal{F}_0|$ on the Cartesian patch of the coordinate system e^a.

We note that, in agreement with (27), the relation (26) may be rewritten in the form

$$\mathbf{D} = \frac{|J|}{|2(\mathcal{F} - \mathcal{F}_0)|^{3/2}}\boldsymbol{\nabla}\mathcal{F} = |J|\boldsymbol{\nabla}\Phi_c, \quad (28)$$

which will be used in what follows. Here,

$$\Phi_c = \frac{1}{|2(\mathcal{F} - \mathcal{F}_0)|^{1/2}}. \quad (29)$$

Now let us show that in a neighborhood of the point P_0, which is the image of the origin of coordinates e^a, Eq. (23) is equivalent to Maxwell's first equation for the induction of a point charge placed at that point. Due to the assumptions made and the properties of Dirac's δ function, in a neighborhood of the point P_0 we have

$$\delta(\mathbf{e}(\mathbf{x})) = \frac{1}{|J|}\delta(\mathbf{x} - \mathbf{x}_0). \quad (30)$$

As a result, Eq. (23) takes the following form:

$$\frac{\partial}{\partial x^\alpha}\left(|J|\frac{E^\alpha}{|2(\mathcal{F} - \mathcal{F}_0)|^{3/2}}\right) = 4\pi\varepsilon\delta(\mathbf{x} - \mathbf{x}_0), \quad (31)$$

or

$$\operatorname{div}\mathbf{D} = 4\pi\varepsilon\delta(\mathbf{x} - \mathbf{x}_0). \quad (32)$$

It is Maxwell's first equation for the electric induction field of a point charge in the background coordinate space. Therefore the induction vector field $\mathbf{D}$ near the point $\mathbf{e} = 0$ can be presented in the form

$$\mathbf{D} = |J|\frac{E^\alpha}{|2(\mathcal{F} - \mathcal{F}_0)|^{3/2}} = \frac{\varepsilon\mathbf{r}}{|\mathbf{r}|^3} + \operatorname{rot}\mathbf{h}, \quad (33)$$

where $\mathbf{h}$ is a certain smooth vector field and $\mathbf{r} = \mathbf{x} - \mathbf{x}_0$. It means that the form of the induction field near a point charge in the background space differs from the strict Coulomb law by the additional field $\operatorname{rot}\mathbf{h}$ whose divergence strictly vanishes, which in classical electrodynamics is treated as the existence of a magnetic field.

The above constructions show that Eqs. (19) and their integral (21) connect the elements of the analytic structure of the function $\mathcal{F}$—its extrema—with point electric charges whose induction field satisfies the first equation of the Maxwell theory. According to (21), the physical markers to which the coordinate system e^a is attached in this approach are the extrema of the function $\mathcal{F}$, perceived as point charges, and its isosurfaces which, according to (21), are mapped into spheres of finite radius. Since the function $\mathcal{F}$ is interpreted as an element of the external geometry of the physical space, namely, the height function, we obtain the desirable construction for building a model of quantum mechanics described in the first sections of this paper. Let us note that in principle, in the framework of this theory, it is possible to consider non-isolated extrema of the function $\mathcal{F}$. Such structures will create non-point singular sources of the electric field which emerge, for example, in algebraic-geometric theories (see, e.g., [17, 18]). However, in the framework of this approach it is possible not to consider such structural elements because non-point extrema and saddle points are topologically unstable and are removed by infinitesimal wigglings (perturbations) of the function $\mathcal{F}$ [11, 16], since the Morse functions form an everywhere dense set of functions in the space of smooth function.

7. CRITICAL POINTS OF A SMOOTH FUNCTION AND POINT CHARGES

The construction connecting the coordinates e^a with elements of the analytic structure of the fundamental potential $\mathcal{F}$ has been so far determined separately for the neighborhood of its each extremum. To build a full description of "particle" dynamics in the whole physical space, it is necessary to build an extension of this construction to the whole physical space V^3 and background space $\mathcal{V}^3$.

Let now the function $\mathcal{F}$, as a Morse function, have extrema at the points P_i with the coordinates $\mathbf{x}_i$, at which all its first-order derivatives turn to zero:

$$\left.\frac{\partial \mathcal{F}}{\partial x^\alpha}\right|_{P_i} = 0. \tag{34}$$

Among the points where the conditions (34) hold are also saddle points, to be denoted S_i. All points where the conditions (34) hold are conventionally called critical points. By definition, a Morse function, and $\mathcal{F}$ in particular, possesses the property that its any critical point (including saddle points) has a neighborhood where other critical points are absent [16, 19]. The saddle points and extrema in Morse theory [16, 19] can be distinguished using the Morse index m. By definition, the Morse index is equal to the number of negative eigenvalues of the matrix of second-order derivatives of this function at the corresponding critical point, i.e., the matrix

$$M^{(i)}_{\alpha\beta} = \left.\left(\frac{\partial^2 \mathcal{F}}{\partial x^\alpha \partial x^\beta}\right)\right|_{\mathbf{x}=\mathbf{x}_i}.$$

For a minimum $m = 0$, for a maximum $m = 3$, and for two types of saddle points, accordingly, $m = 1$ or 2.

By the constructions carried out in the previous section, in a neighborhood of each extremum of the function $\mathcal{F}$ located at a point P_i, there is a coordinate system $e^a(\mathbf{x}, t)$ satisfying Eq. (19):

$$\frac{\partial x^\alpha}{\partial e^a}\frac{\partial \mathcal{F}}{\partial x^\alpha} = \varepsilon_i e^a, \tag{35}$$

where ε_i are calculated according to the rule

$$\varepsilon_i = \operatorname{signdet}\left.\left(\frac{\partial^2 \mathcal{F}}{\partial x^\alpha \partial x^\beta}\right)\right|_{P_i}, \tag{36}$$

similar to (20). Then in this neighborhood of each point P_i the following relation holds:

$$\mathcal{F} = \varepsilon_i \frac{1}{2}|\mathbf{e}|^2 + \mathcal{F}_i, \tag{37}$$

where $\mathcal{F}_i = \mathcal{F}(\mathbf{x}_i)$ is the value of the function $\mathcal{F}$ at the extremum. In the same neighborhood for the field $\mathbf{D}$

$$\mathbf{D} = \frac{|J|}{|\mathbf{e}|^3}\boldsymbol{\nabla}\mathcal{F}, \tag{38}$$

holds the relation (32). If we unify these relations for all neighborhoods of the points P_i, we can write:

$$\operatorname{div}\mathbf{D} = 4\pi \sum_i \varepsilon_i \delta(\mathbf{x} - \mathbf{x}_i). \tag{39}$$

To solve the problem of extending the connection between $\mathcal{F}$ and e^a to the whole space P^3 and accordingly V^3, it is above all necessary to find out how far one can continuously extend the neighborhood of each extremum P_i in which these connecting relations hold without a discontinuity of $e^a(\mathbf{x}, t)$. This can be done by analyzing the relations (21) near each extremum. Each extremum of $\mathcal{F}$, with the value of $\mathcal{F}$ equal to $\mathcal{F}_i$, is mapped into the origin of a Cartesian patch of the coordinate system e^a in such a way that all isosurfaces of the function $\mathcal{F}$, located near this extremum, are mapped into concentric spheres centered at the origin (Fig. 1). Therefore, the mapping $\Im$: $\mathbf{x} \to \mathbf{e}$ is many-sheeted. Since each isosurface of $\mathcal{F}$ is mapped into a sphere of radius $R_i(\mathcal{F}) = 2|\mathcal{F} - \mathcal{F}_i|$ on the Cartesian patch of the coordinate system e^a, it remains to solve the question of the existence, for each extremum, of a limiting isosurface mapped to a sphere. Such a peculiar isosurface, according to Morse's theory [16, 19], can be either an isosurface containing at least one saddle point (see Fig. 2) or an isosurface containing an infinitely remote point. By Morse's theory, the peculiar isosurface bounds a domain that represents a disconnected union of cells, i.e., open sets, topologically diffeomorphic to a ball of finite radius.

There is precisely one extremum in every such cell. Hence it follows that if we exclude from our analysis the peculiar isosurfaces containing an infinitely remote point, then the whole remaining space splits into cells bounded by peculiar isosurfaces, and in each cell one can specify in a continuous manner a coordinate system e^a for which the connection equations (19) and (21) are valid.

We would like to note that domains bounded by peculiar isosurfaces with an infinitely remote point due to the properties of a Morse function could be included in the general scheme under consideration. Such a situation emerges if the function $\mathcal{F}$ has only two extrema, a maximum and a minimum. This can be done if we add an infinitely remote point to the space P^3, which is considered to be Euclidean, and accordingly to V^3. This procedure is called compactification [12]. In this case the isosurface containing

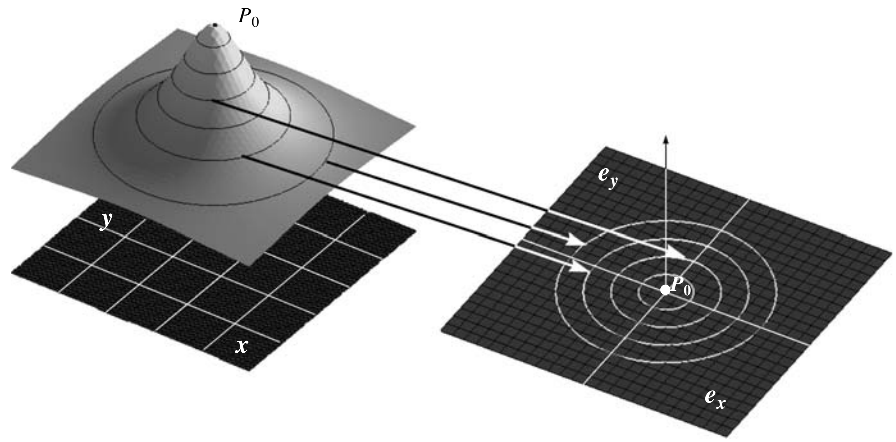

Fig. 1. A two-dimensional analogue of mapping the isosurfaces of the function $\mathcal{F}$ to concentric circles on the Cartesian patch $\mathbf{e}$. P_0 is the extremum and its image on the patch $\mathbf{e}$.

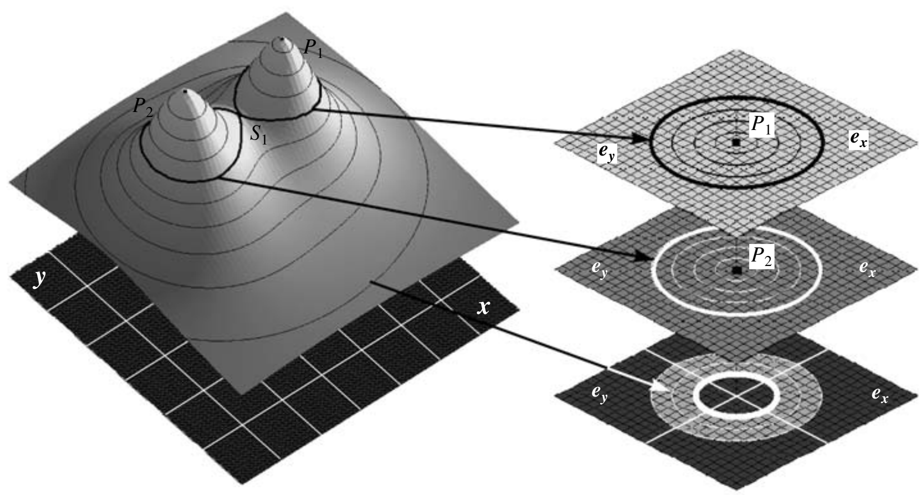

Fig. 2. A two-dimensional analogue of the function $\mathcal{F}$ with a saddle point S_1. P_1 and P_2 are extrema lying inside the cells bounded by the peculiar isosurface. The latter is depicted by a thick line.

the infinitely remote point either turns into a two-dimensional sphere, as it happens in the above case with two extrema, a maximum and a minimum of $\mathcal{F}$, or into a peculiar hypersurface with at least one saddle point. However, for most of the conclusions of this paper, a consideration of isosurfaces with an infinitely remote point is unnecessary. Besides, we will further on give one more simple solution to the problem of the infinitely remote point. Therefore we will not discuss it in detail here.

8. PECULIAR ISOSURFACES AND POINT CHARGES

To complete the extension, it remains to answer the question of how to modify the right-hand side of Eq. (39) on peculiar isosurfaces so that this equation make sense everywhere on P^3 and V^3. This problem can also be solved in another way, by introducing some universal boundary conditions on peculiar isosurfaces, using which one could match separate coordinate patches e^a specified in the neighborhood of each extremum by the relations (19) and (21). However, the question of making more precise the right-hand side of Eq. (39) is extremely important since it indicates the actual charge value in any domain, including peculiar isosurfaces and saddle points.

As an evident example, let us consider the domain $\mathcal{V}_i$ containing two extrema at points P_1 and P_2 and bounded by the peculiar isosurface of the function $\mathcal{F}$. On this isosurface there is the saddle point S_1. A two-dimensional analogue of this structure of the physical space is presented in Fig. 2. Let the function $\mathcal{F}$ take

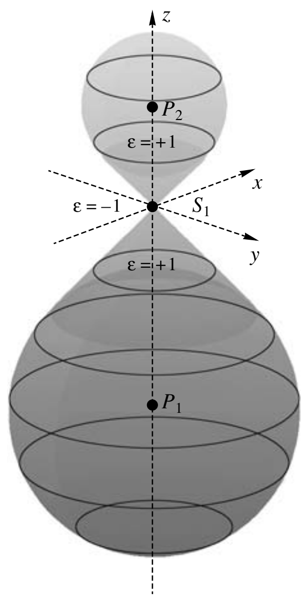

Fig. 3. A view of a peculiar isosurface near a saddle point.

the value $\mathcal{F}_i^{(s)}$ on this peculiar isosurface. A typical example of such a domain is presented in Fig. 3.

Consider Eq. (19) in a neighborhood of the saddle point. By definition, at this point all first-order derivatives of the function $\mathcal{F}$ turn to zero. Consequently, if the Jacobi matrix of the mapping $\Im : \mathbf{x} \to \mathbf{e}$ is not degenerate, then in the limit of the point S_i we have

$$\lim_{\mathbf{x}\to\mathbf{x}(S_i)} e^a = \varepsilon_i \lim_{\mathbf{x}\to\mathbf{x}(S_i)} \left(\frac{\partial x^\beta}{\partial e^a}\frac{\partial \mathcal{F}}{\partial x^\beta}\right) = 0.$$

Here $\mathbf{x}(S_1)$ are coordinates of the point S_1 in $\mathcal{V}_i$. Hence it follows that extending Eq. (19) to the saddle point, we must take into account that in the right-hand side of the identity (22), after its transformation to the coordinates of the physical space, an additional source must appear in the form of a delta function because at that point $e^a(\mathbf{x}(S_i), t) = 0$. Indeed, the right-hand side of Eq. (22) does contain the delta function $\delta(\mathbf{e})$. Therefore in a neighborhood of a saddle point which does not contain other critical points, we have

$$\delta(\mathbf{e}(\mathbf{x}, t)) = \frac{1}{|J(\mathbf{x}_i(S_i))|}\delta(\mathbf{x} - \mathbf{x}(S_i)).$$

However, there emerges a question: which sign $\varepsilon(S_1)$ is it necessary to choose in Eq. (19) near this saddle point? If one approaches it from inside the domain $\mathcal{V}_i$, from the extremum P_1 or P_2, then, as is shown in Fig. 3, it will be the sign $\varepsilon = +1$. The choice of the sign is conditional and corresponds to the minima of $\mathcal{F}$ at points P_1 and P_2. If one approaches S_1 from outside, then the sign ε will be the opposite, $\varepsilon = -1$. Hence it follows that in a neighborhood of the saddle point S_1 it is necessary to write the constraint equation (19) in the following form:

$$\frac{\partial x^\beta}{\partial e^a}\frac{\partial \mathcal{F}}{\partial x^\beta} = \varepsilon_a e^a, \tag{40}$$

where the choice of a sign for each index is made separately and is determined depending on along which coordinate the function $\mathcal{F}$ in the coordinates e^a, approximating the point $e^a = 0$, reaches its maximum or minimum. As a result, an integral of Eq. (40) can be presented in the following form:

$$\mathcal{F} = \varepsilon_i \frac{1}{2}\left((e^1)^2 - (e^2)^2 - (e^3)^2\right) + \mathcal{F}_i^{(s)}. \tag{41}$$

Here $\varepsilon_i = \pm 1$, depending on the Morse index m of the saddle point, namely, $\varepsilon_i = (-1)^m$. Meanwhile,

$$\varepsilon_j = \text{signdet}\left(\frac{\partial^2 \mathcal{F}}{\partial x^\alpha \partial x^\beta}\right)\bigg|_{S_j}. \tag{42}$$

Using these notations, we verify that the relations (19) and (40) as well as (21) and (41) have a common form.

To find the final form of Eq. (39) in the whole background space, taking into account the existence of saddle points, it is necessary to consider the boundary conditions for the field $\mathbf{D}$ of peculiar isosurfaces. As has been shown, at a saddle point itself the structure of the mapping is such that in the background space this point, just as an extremum, corresponds to a point source, a point charge. The sign and value of this charge are determined by (42). As to the common charge, it is determined in this theory by the boundary conditions for the field $\mathbf{D}$ on peculiar isosurfaces because, from a physical viewpoint, the discontinuity of the normal component of the field $\mathbf{D}$ on a peculiar isosurface is equivalent to the appearance of a surface charge. Such charges are only an artifact of the mathematical description. In the case of continuity of the field $\mathbf{D}$ on the peculiar isosurfaces, we exclude the nonphysical surface charges from the theory and provide the fulfilment of the observed integer character of the particles' electric charges. Due to the definition (38) as well as smoothness of $\mathcal{F}$ and orthogonality of the gradient $\boldsymbol{\nabla}\mathcal{F}$ to the peculiar isosurface, the continuity condition for $\mathbf{D}$ on this surface reduces to the condition

$$\left(\frac{|J|}{|\mathbf{e}(\mathbf{x}, t)|^3}\right)\bigg|_{\partial\mathcal{V}_i+0} = \left(\frac{|J|}{|\mathbf{e}(\mathbf{x}, t)|^3}\right)\bigg|_{\partial\mathcal{V}_i-0}, \tag{43}$$

where $\partial\mathcal{V}_i + 0$ means the limit from outside to points of a peculiar isosurface which is a boundary of the domain $\mathcal{V}_i$, and $\partial\mathcal{V}_i - 0$ is the same from inside. The validity of the continuity condition (43), assuring the absence of surface charges on peculiar isosurfaces, makes it possible to write the following equation as an equation for the function $\mathbf{D}$ in the whole background space:

$$\operatorname{div}\mathbf{D} = 4\pi\sum_i \varepsilon_i\delta(\mathbf{x}-\mathbf{x}_i(P_i)) + \sum_j \varepsilon_j\delta(\mathbf{x}-\mathbf{x}_i(S_j)). \quad (44)$$

It contains only point charges as sources, and their value and sign are determined with the aid of Morse indices of the critical points of the fundamental potential $\mathcal{F}$:

$$\varepsilon_i = (-1)^{m_i}, \quad (45)$$

where m_i is the Morse index of the i-th critical point.

We postpone the analysis and a discussion of the physical meaning of the boundary conditions (43) for afterwards. Let us now only remark that these boundary conditions must be considered together with the continuity condition for the field $\mathbf{V}$ that determines the transport of the coordinate system $e^a(\mathbf{x}, t)$ in the background space. Due to (17), the conditions (43) are reduced to continuity of the ratio of the squared absolute value of the wave function Ψ to the function $|\mathbf{e}(\mathbf{x}, t)|^3 = (2|\mathcal{F}-\mathcal{F}_i|)^{3/2}$, which at different choices of values of $\mathcal{F}_i$ in different domains $\mathcal{V}_i$ experiences a discontinuity at the peculiar isosurface (the function $\mathcal{F}$ remains continuous). This contradicts the continuity postulate for the wave function in quantum theory. Thus to satisfy the continuity condition for the squared absolute value of the wave function it is necessary to make a universal choice of the value of $\mathcal{F}_i$ for all domains $\mathcal{V}_i$.

9. THE ELECTRIC CHARGE, PARTICLES AND TOPOLOGY

Let us now consider more deeply the consequences of Eq. (44) derived in the previous section. Among them, the most important one is the topological nature of the electric charge value corresponding to this equation. Integrating (44) over some domain $\mathcal{V}$ of the background space, we obtain the relation

$$Q = \frac{1}{4\pi}\int_{\mathcal{V}} \operatorname{div}\mathbf{D}\, d\mathcal{V} = \frac{1}{4\pi}\int_{\partial\mathcal{V}} (\mathbf{D}, \mathbf{n})\, d\sigma = \sum_{i=1}^{N} (-1)^{m_i}, \quad (46)$$

where the sum in the right-hand side is taken over all critical points of the function $\mathcal{F}$. The quantity Q is by this definition the value of the total charge concentrated in the domain $\mathcal{V}$. Using (36) and (42), we obtain for the electric charge Q the following expression:

$$Q = \sum_{i=1}^{N} \operatorname{signdet}\left(\frac{\partial^2\mathcal{F}}{\partial x^\alpha \partial x^\beta}\right)\Bigg|_{P_i}, \quad (47)$$

where the sum is taken over all critical points of the function $\mathcal{F}$. The sum in the right-hand side is a topological invariant [20] of the vector field $\mathbf{E} = \boldsymbol{\nabla}\mathcal{F}$, equal to a half of the index of this field on the surface $\partial\mathcal{V}$ that bounds the domain $\mathcal{V}$. If the vector field $\mathbf{E}$ is transversal to the surface $\partial\mathcal{V}$ (see [20]), i.e., it is everywhere on this surface directed either inside the domain $\mathcal{V}$ or outside it, then, according to the Poincaré—Hopf index theorem, the number Q is equal, up to the sign, to the Euler characteristic $\chi(\mathcal{V})$ of the domain $\mathcal{V}$ [11, 20, 21]. The same result can also be obtained using the Poincaré-Hopf theorem on the index of a vector field (in the present case, the field $\mathbf{E}$) on the domain $\mathcal{V}$.

A special case of choosing the domain $\mathcal{V}$ at whose boundary the field $\mathbf{E}$ is transversal to $\partial\mathcal{V}$, more precisely, is orthogonal to it, is a domain bounded by an isosurface of the function $\mathcal{F}$ and, in particular, by a peculiar isosurface. Since peculiar isosurfaces play the role of distinguished surfaces in the background space which split it into separate cells, and in which the dynamics is separated from the viewpoint of the Schrödinger equation, one can assume that it is these surfaces that allow for unambiguously singling out, on geometric grounds, the domains which behave in the experiment as elementary particles. Following this conjecture, **we will understand as an elementary particle a geometrically distinguished spatial domain bounded by a peculiar isosurface of the fundamental potential** $\mathcal{F}$. Since the boundaries of peculiar isosurfaces are distinguished on V^3 by equations of the form

$$\mathbf{e}^2 = R^2 = \text{const},$$

they satisfy the basic requirement formulated in the derivation of the Schrödinger equation (18) for domains to be interpreted as separate quantum particles. Therefore particles according to the notion formulated above automatically acquire the properties of quantum objects whose state is described by the Schrödinger equation.

On the basis of this assumption, let us build a classification of possible topological structures of such domains and connect it with the value of the electric charge to be possessed by such particles. This will give us an opportunity to compare this topological classification with the properties of real particles and establish the ability of the present approach to explain the laws observed in experiments with real particles.

A basis of the classification is given by a formula which is fundamental in this theory, follows from Eq. (47) and, as was explained above, connects the particle charge with the Euler characteristic of domains bounded by peculiar isosurfaces:

$$Q = \varepsilon\chi(\mathcal{V}), \quad (48)$$

where $\varepsilon = \pm 1$ is the sign determined by the direction of the field $\mathbf{E}$ on the surface $\partial\mathcal{V}$ of the domain $\mathcal{V}$: $\varepsilon = +1$ if the field $\mathbf{E}$ on ∂V is everywhere directed outward, and $\varepsilon = -1$ if it is directed inward. The geometric meaning of the charge, if we treat the fundamental potential $\mathcal{F}$ as a function of height of the physical hypersurface in W^4, consists in the direction of "convexity" of the physical space in W^4 (see Fig. 5) with respect to the background hypersurface. The Euler characteristic of n-dimensional manifolds M^n is determined by the formula

$$\chi(M^n) = \sum_{i=1}^{n}(-1)^i p_i, \quad (49)$$

where p_i are the dimensions of the homology groups of the manifolds M^n of the i-th dimension (or Betty number) [12, 20]. This integer-valued quantity is a topological invariant, i.e., does not depend on smooth deformations of the manifold M^3. Our interest will be in two-dimensional manifolds, isosurfaces of the fundamental potential, and in three-dimensional domains bounded by these isosurfaces.

Isosurfaces of a Morse function (possibly after compactification) are [16, 19] closed (having no boundaries) two-dimensional orientable (two-sided) manifolds. All such isosurfaces are, up to smooth deformations, equivalent to a two-dimensional sphere with g handles [11, 12]. In agreement with this classification, the Euler characteristics of all closed, two-dimensional, compact, orientable surfaces can be calculated by the formula

$$\chi(\sigma_g^2) = 2(1-g). \quad (50)$$

Here we have $p_0 = 1$, $p_1 = 2g$, $p_2 = 1$ [11]. Thus, for a two-dimensional sphere $\sigma_0^2 = \mathcal{S}^2$, and for it the number g is zero, $\chi(\sigma_0^2) = 2$; for a torus which is by definition equivalent to a sphere with the number of handles $g = 1$, $\chi(\sigma_1^2) = 0$, for a "krendel" (knot-shaped, or eight-shaped, biscuit) with $g = 2$, $\chi(\sigma_2^2) = -2$ and so forth.

There is a general relation [11, 20] that connects the Euler characteristics of two-dimensional closed orientable surfaces σ with the Euler characteristics of domains $\mathcal{V}$ whose boundaries are the surfaces $\sigma = \partial\mathcal{V}$:

$$\chi(\mathcal{V}) = \frac{1}{2}\chi(\partial\mathcal{V}). \quad (51)$$

Therefore the charge value corresponding to (48) in a domain bounded by an isosurface of the function $\mathcal{F}$ is completely determined by the structure of the domain boundary, i.e., by the number g of its handles and may be calculated by the formula

$$Q = \frac{\varepsilon}{2}\chi(\partial\mathcal{V}) = \varepsilon(1-g). \quad (52)$$

For a domain bounded by a sphere, the charge can be equal to ± 1: $Q = \pm 1$; for a domain bounded by a torus, the charge can be zero, $Q = 0$; for an eight-shaped domain $Q = \mp 1$, etc. Thus the charge of any particle is completely determined by the structure of its peculiar isosurface.

10. CLASSIFICATION OF PARTICLES

The structure of particles, according to their definition, can be described as the structure of domains bounded in the background space by peculiar isosurfaces of the fundamental potential. A peculiar isosurface is not smooth since it contains saddle points (at least one) at which the isosurface has a conical singularity (see Fig. 3). Therefore, to classify the structure of a domain $\mathcal{V}_i$, it is more helpful to use, instead of its boundary, an infinitely close isosurface of the fundamental potential which is smooth, oriented and can always be chosen as a compact one (maybe with the aid of the compactification procedure).

Let the value $\mathcal{F}_s$ of the fundamental potential correspond to the peculiar isosurface $\partial\mathcal{V}_i$ that bounds a certain domain cV_i. Let us denote by $\mathcal{W}_i^+(\delta\mathcal{F})$ and $\mathcal{W}_i^-(\delta\mathcal{F})$ the domains with the boundaries $\partial\mathcal{W}_i^+(\delta\mathcal{F})$ and $\partial\mathcal{W}_i^-(\delta\mathcal{F})$, which are isosurfaces of the fundamental potential with its values $\mathcal{F}_s + \delta\mathcal{F}$ and $\mathcal{F}_s - \delta\mathcal{F}$, respectively, where $\delta\mathcal{F} > 0$ is any nonnegative infinitesimal number. Since $\mathcal{F}$ is a Morse function, all its critical points are isolated. It means that for any peculiar isosurface one can always find such a nonnegative, sufficiently small number $\delta\mathcal{F}$ that the surfaces $\partial\mathcal{W}_i^+(\delta\mathcal{F})$ and $\partial\mathcal{W}_i^-(\delta\mathcal{F})$ do exist, are smooth and orientable. Since only a small number of isosurfaces of a Morse function can contain an infinitely remote point, one can always choose the quantity $\delta\mathcal{F}$ in such a way that the isosurfaces $\partial\mathcal{W}_i^\pm(\delta\mathcal{F})$ will be compact and will not contain an infinitely remote point. Thus we can always get rid of domains with an infinitely remote point. Therefore, discussing further the particle classification by structure of the domains $\mathcal{V}_i$, we will assume not this domain itself but rather the domain $\mathcal{W}_i^\pm$ that completely contains the domain $\mathcal{V}_i$. One of the domains, either $\mathcal{W}_i^+$ or $\mathcal{W}_i^-$, will always meet this requirement. For convenience, we will omit the sign index in the notation of this domain.

As has been noted, to calculate the particle charge it is sufficient to know the structure of the boundary of the domain $\mathcal{W}_i$. However, it is insufficient for describing the inner structure of the domains $\mathcal{W}_i$. The topological structure of three-dimensional manifolds does not have such a simple classification as that of two-dimensional manifolds. However, for the purposes of the present paper, it will be sufficient to restrict ourselves to domains that can be obtained from a three-dimensional ball or any other simply connected domain $\mathcal{W}_i$, bounded by an isosurface of the function $\mathcal{F}$, with the aid of a simple topological restructuring: joining a three-dimensional handle. Such a modification, by analogy with the two-dimensional case (see [11]), consists in that from the 3-domain $\mathcal{W}_i$ one cuts out two balls and, by edges of these balls, joins to them a three-dimensional cylinder. The obtained construction is called a three-dimensional handle or a Wheeler handle. Such domains are not simply connected. Repeating this procedure b times, we obtain a domain $\mathcal{W}_i$, having b Wheeler handles. The idea of using such domains for a description of particle structure has been put forward in [8–10]. Following this idea, to describe a particle it is necessary to indicate two basic numbers: g, the number of handles of the boundary of $\mathcal{W}_i$, and the number b of handles "glued" into this domain. The sign of the baryonic charge, equal by absolute value to the number b of Wheeler handles, is determined by the side with respect to the background space from which a handle has been glued (see Fig. 5). For completeness of the description of all possible situations besides the indicated domain structure types, one should mention the opportunity that the domain $\mathcal{W}_i$ can itself include a certain number of similar domains $\mathcal{W}_{i,m}$: $\mathcal{W}_{i,m} \subset \mathcal{W}_i$.

Introducing into consideration domains with Wheeler handles, we can now improve the formula for a particle's charge. It has been noted that the charge value is determined by the number g of handles of the boundary of the domain $\mathcal{W}_i$, however, the definition of the Euler characteristic also contains, according to (49), the Betty numbers p_i, $i = 0, 1, 2, 3$, whose values are, apart from this, determined by the number of Wheeler handles. For the considered case of 3D compact orientable domains with the boundary $\partial\mathcal{W}_i$, we have for Betty numbers [12]:

$$p_0 = p_3 = 1, \quad p_1 = b + g, \quad p_2 = 1 + b. \qquad (53)$$

Substituting these values to the general formula (49), we find:

$$\chi(\mathcal{W}_i) = p_0 - p_1 + p_2 - p_3 = 1 - g, \qquad (54)$$

or

$$Q = \varepsilon(p_0 - p_1 + p_2 - p_3) = \varepsilon(p_2 - p_1) = \varepsilon(1 - g). \qquad (55)$$

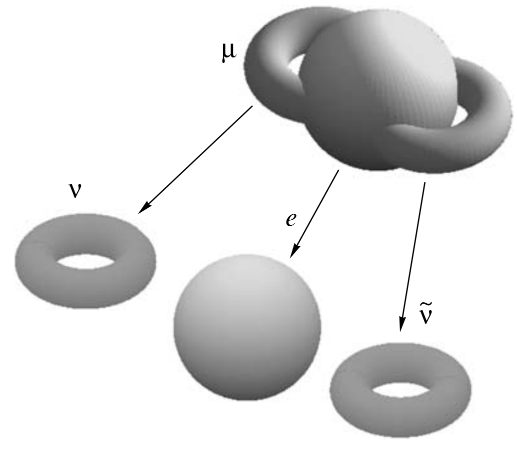

Fig. 4. Topological reconstruction of muon decay.

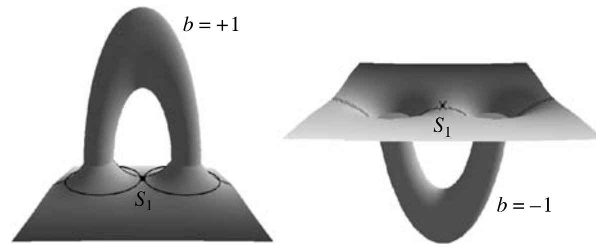

Fig. 5. A two-dimensional analogue of the geometric structure of Wheeler handles corresponding to the proton and the antiproton. S_1 is a saddle point. The solid curve indicates the external peculiar isosurface.

Having introduced non-simply connected domains like Wheeler handles into consideration, we fill the gap in justifying the function $\mathcal{F}$ as a height function of the physical hypersurface embedded in the ambient space W^4. At the beginning of the paper we have referred to the works [8–10] where this problem was discussed. Now we can formulate here as well the necessity of precisely this interpretation of $\mathcal{F}$ from a physical viewpoint. The point is that when there are multiply connected domains projected onto a simply connected Euclidean space, there emerges the problem of interpretation of simultaneous existence of a few independent "copies" of the same physical quantity, such as, for instance, the electric field strength. The only non-contradictory explanation of this situation can be the explanation that it is the real physical space that possesses a complicated topology rather than the field function.

10.1. Leptons

Leptons are the simplest and the lightest particles, of which two, the electron and the neutrino, are stable. The latter circumstance can be considered as an indication of the maximum elementarity of such

Topological classification of leptons

Particle	Number of handles of the boundary	Charge
electron $e^{\pm}$:	$g = 0$	$Q = \pm 1$
neutrino ν:	$g = 1$	$Q = 0$
muon $\mu^{\mp}$:	$g = 2$	$Q = \mp 1$

particles: there is nothing to which they might decay. Therefore, for their classification, the simplest structures corresponding to $b = 0$ can be used. Then the classification of leptons can be formulated on the basis of the structure of boundaries $\partial\mathcal{W}_i^{\pm}$ and should look as follows:

The corresponding structures are presented in Fig. 4. This interpretation allows for the assumption that there can be leptons in the nature with charges larger than 1. Such particles would correspond to structures with $g > 2$. These particle should be unstable and decay for a much smaller time than that of muon decay. The main decay channel of a muon is $\mu \to e + \nu + \tilde{\nu}$, and its topological reconstruction is given in Fig. 4.

Fig. 4 demonstrates that, in the framework of the suggested classification of particles, the muon decay looks as the process of a transformation of the domain boundary and formation of several separate peculiar isosurfaces whose structure precisely corresponds to the initial muon structure. Each handle of the initial surface forms its own neutrino. Decays of leptons with charges larger than 1 should occur according to similar scenarios, with a topological transformation of the domain boundary and emergence of new peculiar isosurfaces with neutrino structure. The number of neutrino after a decay of a lepton with the charge $|Q| = |1 - g| > 1$ should be equal to g and the number of electrons (positrons) equal to $|Q|$.

10.2. Baryons

One more class of particles some of which are stable are the baryons. Among stable baryons, the simplest one is the proton, forming the nucleus of a hydrogen atom. Other stable baryons are nuclei of heavier atoms. A distinction of the baryons from the leptons is that their mass is three orders of magnitude larger and that they take part in the very "intense" and short-range interaction, the strong, or nuclear one. Among the topological structures suitable for assigning to observable particles are structures with a given number of Wheeler handles. The latter are, by construction, localized non-simply connected geometric objects possessing a very large local curvature. If, following Einstein's general relativity, we believe that the curvature determines the energy of interaction, then one can assume that the existence of Wheeler handles in the baryon structure is just the key feature in explaining the nuclear interaction forces. Therefore, in the framework of the suggested conception, it is natural to connect the number of Wheeler handles with the baryonic charge of these particles. Then among all domains with Wheeler handles in their structure, the simplest is the one with a single Wheeler handle, i.e., with $b = 1$ and a sphere-shaped external boundary ($g = 0$) (see Fig. 5).

Since the neutron is practically the same as the proton by its internal structure but has a zero electric charge, it is easy to assume that the neutron should correspond to a domain with $b = 1$ and $g = 1$, i.e., such a domain has a Wheeler handle in its structure while the domain boundary is a torus. The neutron is a quasistable particle whose decay in its free state occurs by the main channel $n \to p + e + \nu$. A topological reconstruction of this channel again corresponds to a transformation of the domain boundary with "separation" on one handle from it (the neutrino) and conversion of the boundary to the simplest one, the sphere $\mathcal{S}^2$. It is hard to represent this reconstruction graphically due to the 4D nature of such a transformation. The neutron decay is connected with the weak interaction. It is the same interaction that is related to muon decay. It enables us to assume that, in the framework of the present approach, the weak interaction is connected with transformations of domain boundaries, i.e., with an increasing or decreasing number g in the boundary structure of the initial particle in the case of its instability.

The above interpretation of particles allows for giving a simple interpretation to the Gell-Mann—Nishijima formula [22]

$$Q = \frac{b + S}{2} + J_3.$$

Here S is the strangeness, J_3 is the isospin projection, and b the baryonic charge. Comparing this formula with (55), we find:

$$\begin{aligned} J_3 &= \varepsilon(p_2 - p_1) - (b + S)/2 \\ &= \varepsilon(1 - g) - (b + S)/2. \end{aligned} \qquad (56)$$

Assuming that the strangeness is zero for the nucleon doublet, we obtain the standard isospin values of the proton and nucleon: $J_3(p) = 1/2$, $J_3(n) = -1/2$. Thus the Gell-Mann—Nishijima formula is not connected with some algebraic symmetries which can in principle be invented in the framework of the theory of homologies and cohomologies of the particle domains $\mathcal{W}_i$, but is a certain special treatment of the Betty numbers.

The presented classification of particles is rather superficial. However, the results obtained in its framework show that it is able to explain some well-known facts of particle physics. This enables us to continue the development of this theory in the direction of building a more general topological theory of particle structure.

11. EQUATIONS OF ELECTRODYNAMICS WITH INTEGER-VALUED CHARGE

Let us now consider the construction of the remaining equations of Maxwell's electrodynamics in the approach under study. We begin with calculating the field $\mathbf{D}$ as a solution to Eq. (44). The general solution to Eq. (44) can be presented in the form

$$\mathbf{D} = J\frac{\boldsymbol{\nabla}\mathcal{F}}{|\mathbf{e}|^3} = 4\pi\sum_{i=1}^{N}\varepsilon_i\frac{\mathbf{r}_i}{|\mathbf{r}_i|^3} + \operatorname{rot}\mathbf{h}, \qquad (57)$$

where $\mathbf{h}$ is a certain smooth vector field, $\varepsilon_i = (-1)^{m_i}$, $\mathbf{r}_i = \mathbf{x} - \mathbf{x}_i$, and $\mathbf{x}_i$ are the radius vectors of the positions of critical points of the fundamental potential. This representation follows from the form of the right-hand side of (44), containing a sum of δ functions at critical points with the coordinates $\mathbf{x}_i$. The right-hand side of (57) is a sum of the classical electric field strength of the point charges $\mathbf{E}_c$,

$$\mathbf{E}_c = 4\pi\sum_{i=1}^{N}\varepsilon_i\frac{\mathbf{r}_i}{|\mathbf{r}_i|^3}, \qquad (58)$$

and the divergence-free field $\operatorname{rot}\mathbf{h}$. This allows for identifying the field $\mathbf{D}$ with the strength $\mathbf{E}$ of the classical electric field if the field $\operatorname{rot}\mathbf{h}$ is represented as a time derivative of a vector potential:

$$\operatorname{rot}\mathbf{h} = \frac{1}{c}\frac{\partial\mathbf{A}}{\partial t}. \qquad (59)$$

Here c is a certain constant having the meaning of the speed of light in classical electrodynamics. In this case, we have the relation

$$\mathbf{E} = \mathbf{D} = \mathbf{E}_c + \frac{1}{c}\frac{\partial\mathbf{A}}{\partial t}.$$

This relation means that the vacuum dielectric permittivity is equal to 1. The field $\mathbf{E}_p$ is a potential one:

$$\mathbf{E}_c = -\boldsymbol{\nabla}\Phi,$$

where

$$\Phi = -4\pi\sum_{i=1}^{N}\frac{\varepsilon_i}{|\mathbf{r}_i|}.$$

The result of time differentiation of (57) can be interpreted as one more Maxwell equation:

$$\frac{\partial\mathbf{D}}{\partial t} = -4\pi\mathbf{j} + c\operatorname{rot}\mathbf{H}, \qquad (60)$$

if one takes for the current density $\mathbf{j}$ the field

$$\mathbf{j} = -\frac{\partial}{\partial t}\left(\sum_{i=1}^{N}\varepsilon_i\frac{\mathbf{r}_i}{|\mathbf{r}_i|^3}\right) = \frac{1}{4\pi}\frac{\partial}{\partial t}\boldsymbol{\nabla}\Phi, \qquad (61)$$

and as the magnetic field strength the quantity

$$\mathbf{H} = \frac{1}{c}\frac{\partial\mathbf{h}}{\partial t}. \qquad (62)$$

Taking a curl of both sides of (57), we arrive at the following equation:

$$\operatorname{rot}\mathbf{D} = \operatorname{rot}\mathbf{E} = \operatorname{rot}\operatorname{rot}\mathbf{h} = \frac{1}{c}\frac{\partial\operatorname{rot}\mathbf{A}}{\partial t}. \qquad (63)$$

Let us introduce the magnetic induction according to the relation $\mathbf{B} = \operatorname{rot}\mathbf{A}$. This automatically gives the third Maxwell equation

$$\operatorname{div}\mathbf{B} = 0. \qquad (64)$$

The last, fourth Maxwell equation is obtained from (63) by the corresponding substitution:

$$\operatorname{rot}\mathbf{D} = \frac{1}{c}\frac{\partial\mathbf{B}}{\partial t}. \qquad (65)$$

Then the magnetic field strength $\mathbf{H}$ and induction $\mathbf{B}$ are connected by the common equation

$$\frac{1}{c^2}\frac{\partial^2\mathbf{B}}{\partial t^2} = \operatorname{rot}\operatorname{rot}\mathbf{H}. \qquad (66)$$

This equation is in fact an identity created by the existence of the field $\mathbf{h}$ in the theory. The identity (66) can be verified by a direct substitution of the relations (59) and (62) into (66) together with the definition $\mathbf{B} = \operatorname{rot}\mathbf{A}$. So this equation does not impose any restrictions on the form of the field $\mathbf{h}$ and only expresses the general form of a relationship between the fields introduced to the theory for their confrontation with classical electrodynamics. Therefore this relation in the present theory does not contradict the material equation of classical electrodynamics $\mathbf{B} = \mu_0\mathbf{H}$, where μ_0 is the magnetic permeability of the vacuum. One can verify that this material equation is in the present theory equivalent to an equation for the field $\mathbf{h}$ which has the form

$$\operatorname{rot}\operatorname{rot}\mathbf{h} = \frac{\mu_0}{c^2}\mathbf{h}_{tt}. \qquad (67)$$

It is in essence the wave equation of classical electrodynamics with a constant speed of light in vacuum equal to c if $\mu_0 = 1$.

The whole set of identities obtained from the initial relation (44) completely reproduces the form of all Maxwell equations and contains, instead of the standard material equation connecting $\mathbf{B}$ and $\mathbf{H}$, a more general equation, (66), which is equivalent to the introduction of the field $\mathbf{h}$ in agreement with (59) and (62). The constant c was introduced there formally and acquires the meaning of the speed of light if one puts $\mu_0 = 1$, which entirely agrees with classical electrodynamics and corresponds to a special choice of measurement units.

The last important test of the opportunity to put into correspondence the constructed vacuum "microscopic" electrodynamics with classical electrodynamics is the existence of the differential conservation law of charge in the theory. As is easily verified, in the present theory this differential conservation law is obtained directly from the definition of the current density (61) and Eq. (44):

$$\frac{\partial \rho}{\partial t} + \operatorname{div} \mathbf{j} = 0, \tag{68}$$

where the charge density ρ is determined by the right-hand side of Eq. (44).

Thus the present theory preserves all main features of classical electrodynamics provided that the vacuum obeys the material equation $\mathbf{H} = \mathbf{B}$. In classical electrodynamics this relation is a postulate reflecting the experimental fact of the wave nature of electromagnetic signals and the lack of dispersion properties of vacuum. In this theory we can also use this postulate at microscales. However, at small distances corresponding to particle sizes, this requirement could be violated. A reason for such a conclusion is the following important distinction of the present theory from classical electrodynamics.

The emergence of Eq. (66) in the theory is explained by the fact that the electric induction field $\mathbf{D}$ contains from the very beginning the divergence-free term $\operatorname{rot} \mathbf{h}$, which is in general nonzero even in the static case. The latter means that due to (62) point charges possess a static magnetic field. By modern views, a static magnetic field of point particles as quantum objects is connected with the existence of their own angular momentum, the spin. Therefore the emergence of a static magnetic field component at microscales can be considered as an explanation of the spin in the present theory. However, a static magnetic field component cannot emerge in the framework of classical electrodynamics, which requires a change in the material equation $\mathbf{B} = \mathbf{H}$ equivalent to (67), at small scales.

12. CONCLUSION

The obtained topological interpretation of the electric charge makes it possible to approach in a new way the foundation of quantum laws. The basic element of quantum mechanics is the statistical postulate, reflecting the specific behavior of quantum objects that is usually interpreted as a random behavior. The conception built in the present paper is deprived of such a statistical nature. Nevertheless, this conception allows for introduction of such ways of describing the particle dynamics in the interpretation presented above that reproduce all basic features of the statistical interpretation of quantum mechanics. This was described in the first sections of this paper devoted to a derivation of the Schrödinger equation. As was shown, the Jacobian of the transformation connecting the coordinates of the background space with those of geometric markers almost completely reproduces the basic elements of the statistical description of quantum mechanics which now acquires a geometric meaning. This approach as a whole allows for a rational treatment of some laws of quantum mechanics.

As has been pointed out in this paper, the quantum-mechanical laws in their usual standard form work in each separate cell bounded by a peculiar isosurface. Meanwhile, the averaged equations describe the dynamics of cells themselves. The standard approach to the analysis of quantum systems rests on the set of boundary conditions which are necessary for obtaining unique solutions of the Schrödinger equation. These conditions are imposed at infinity. However, in the present approach the boundary conditions for the Schrödinger equation are formulated on peculiar isosurfaces of the fundamental potential. It is therefore important to state that the standard quantum mechanics is reproduced in the framework of the present theory if the peculiar isosurface on which the quantum dynamics is considered has a macroscopic size, which in usual quantum mechanics corresponds to infinitely remote boundaries. For a full foundation of the quantum status of the present theory, it is still necessary to solve some general problems. However, for some of them it is clear how they can be solved on the basis of the general ideology of this theory. For example, the projection postulate that leads to the most radical deflection of quantum physics from the rational nature of classical physics, declaring an instantaneous shrinkage of the wave function to its final states, should be connected in this theory with topological transformations of the domains bounded by peculiar isosurfaces of the function $\mathcal{F}$. The topological transformations occur instantaneously because for changing a topology it is sufficient to change the structure of a function at a single saddle point. As a result, the topological properties of the whole system instantly change.

The topological nature of the present theory has been shown to allow for building a sufficiently developed particle classification which entirely agrees with the existing data on elementary particle structure. Certainly, the particle structure analysis carried out in this paper is not complete. We here did not address the questions concerning the theory of spin, relativistic effects and, which is the main point, the questions related to particle mass calculations. These questions require a separate consideration out of the scope of the present work, where we have only built the general scheme of the approach to a topological and geometric foundation of quantum mechanics in its non-relativistic form and the related electrodynamics with integer-valued charges. However, one can assert in advance that in this theory there will be no difficulties with obtaining finite particle masses because, from the very beginning, here the electromagnetic singularities are an artifact of the mathematical description of a smooth analytic structure connected with the fundamental potential $\mathcal{F}$. Some tentative considerations about the ways of calculating the masses in the present approach can be found in [10].

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